Lecture 5.

From Lecture 4 notes.

- Prop 3. Pf of (iii).
- Lebesgue Covering Lemma + Pf.
- Thm 2.

Continuity.

Def. (1) Let (X, d), (Ω, ρ) be metric spaces, $f: X \to \Omega$ map.
$\lim_{x \to x_0} f(x) = \omega$ if $\lim_{x \to x_0} \rho(f(x), \omega) = 0$;
equivalently $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $x \in B(x_0, \delta)$
$\Rightarrow f(x) \in B(\omega, \varepsilon)$.

(2) f is cont. at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$, and cont. if cont. at every pt.

Prop 1 TFAE

(i) f is cont.

(ii) $\forall$ open $V \subseteq \Omega$, $f^{-1}(V)$ is open in X.

(iii) Same for closed sets.

Pf. DIY.

composition

Rem. $+, -, \cdot, \circ$ preserves cont.

Def. ③ $f : X \to \Omega$ is *unif. cont.* if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x_0 \in X$, $x \in B(x_0, \delta)$ $\Rightarrow f(x) \in B(f(x_0), \varepsilon)$. I.e., δ depends on ε but not x_0.

Thm 1. Let $f : X \to \Omega$ be cont.

(i) X compact $\Rightarrow f(X) \subseteq \Omega$ compact.

(ii) X connected $\Rightarrow f(X) \subseteq \Omega$ connected.

(iii) X compact $\Rightarrow f$ unif. cont.

Pf. (i) Let $\{H_i\}_{i \in I}$ be open cover of $f(X)$. Then, $G_i = f^{-1}(H_i)$ forms open cover of X. But X compact $\Rightarrow$ $\exists$ finite subcover $G_{i_1}, \dots, G_{i_n}$. We claim $H_{i_1}, \dots, H_{i_n}$ covers $f(X)$. Pick $\omega \in f(X)$ and $x \in G_{i_k}$ s.t. $f(x) = \omega$. But $G_{i_k} = f^{-1}(H_{i_k}) \Rightarrow \omega = f(x) \in H_{i_k}$ as claimed.

(ii) DIY.

(iii) Pick $\varepsilon > 0$. By cont. $\forall x \ \exists \delta_x > 0$ s.t. $x' \in B(x, \delta_x) \Rightarrow f(x') \in B(f(x), \varepsilon/2)$

WTS $\exists \delta > 0$ <u>indep. of x</u>. Well, $\{B(x, \delta_x)\}_{x \in X}$ is open cover of X. By Lebesgue Covering Lemma, $\exists \delta > 0$ s.t. $\forall x$ $B(x, \delta) \subseteq B(x_0, \delta_{x_0})$ some x_0. But then if $x' \in B(x, \delta) \Rightarrow x' \in B(x_0, \delta_{x_0})$, so

$$\rho(f(x'), f(x)) \leq \rho(f(x'), f(x_0)) + \rho(f(x_0), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, this δ will do. $\boxtimes$

Thm 1 has many important Cor's. E.g.

$\left.\begin{array}{l}\text{• Intermediate Value Thm.} \\ \text{• Cont. fcn achieves max/min on} \\ \quad \text{compact subsets.}\end{array}\right\}$ calculus